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Intertemporal Separability in Overlapping
Generations Models

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Abstract

Balasko and Shell have shown that in a pure exchange overlapping generations model with many goods, but a single two period lived Cobb-Douglas consumer in each generation, there is generically a unique perfect foresight price path in which money has no value. If money has value, then generically there is at most one dimension of indeterminacy. Kehoe and Levine have shown that these results do not generalize to models wth many consumers and general preferences. In this paper we show that the crucial assumptions of Balasko and Shell are that each generation has a two period lived representative consumer with a utility function that is separable between periods. Analyzing stationary models, we also show that the Balasko-Shell results continue to hold if the two period lived consumers are "almost" identical and have "almost" intertemporally separable preferences.

Intertemporal Separability in Overlapping Generations Models

by

Timothy J. Kehoe and David K. Levine*

1. INTRODUCTION

Balasko and Shell (1981) have studied a pure exchange overlapping generations model with many goods, but a single two period lived Cobb-Douglas consumer in each generation. They prove that there is generically a unique perfect foresight price path in which money has no value. If money has value, then generically there is at most one dimension of indeterminacy, which can be indexed by a "price of money". Kehoe and Levine (1982a) have shown that these results do not generalize to many consumers with general preferences: There are open sets of economies that exhibit many dimensions of indeterminacy regardless of whether or not money has value. This leads us to pose the question: What special properties of the Balasko-Shell model produce their strong results?

We show that the crucial assumptions of Balasko and Shell are that each generation has a two period lived representative consumer with a utility function that is separable between periods. This is equivalent to assuming that the matrix of derivatives of young peoples' excess demand with respect to prices when they are old (and old peoples' excess demand with respect to prices when they are young) has rank one. The resulting theory can then be

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reduced to the case in which there is one good in each period, where the Balasko-Shell result is well-known.

We also show that the Balasko-Shell results on the dimension of indeterminacy continue to hold if the two period lived consumers are "almost" identical and have "almost" intertemporally separable preferences. This shows that their example is not degenerate, but characterizes an open set of economies. Our results are the strongest possible in the sense that Kehoe and Levine (1982b) have found robust examples of economies with a single three period lived consumer with an additively separable (in fact, CES) utility function that exhibit indeterminacy when money has no value and several dimensions of indeterminacy when it does have value. Consequently, the Balasko-Shell results cannot be further generalized. One important case in which the Balasko-Shell results do not generalize is when a single consumer lives more than two periods. This can always be reduced to the case of several consumers who live only two periods, but unfortunately there is no sense in which these consumers are "almost" identical. Thus, as the example in Kehoe and Levine (1982b) shows, the assumption of two periods of life is a severe one, at least when it is coupled with the assumption of a representative consumer.

We state and prove our results locally near the steady state of a stationary economy using techniques quite different from Balasko and Shell. We do, however, indicate how the results with an exactly representative consumer and exactly separable preferences extend globally to non-stationary settings.

2. THE OVERLAPPING GENERATIONS MODEL

Each generation $t > 1$ is identical and lives in periods t and $t + 1$. There are n goods in each period. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ when young and $z(p_t, p_{t+1})$ when old. The vector $p_t = (p_t^1, \ldots, p_t^n)$ denotes prices in period t. Intertemporal trade is possible, so the aggregate budget constraint (Walras's law) has the form $p_t'y(p_t, p_{t+1}) + p_{t+1}'z(p_t, p_{t+1}) = 0$. In addition, excess demand is homogeneous of degree zero in prices (p_t, p_{t+1}). We also assume excess demand is continously differentiable, which, as Debreu (1972) and Mas-Colell (1974) have shown, entails little loss of generality. An equilibrium price path for this economy is one in which excess demand vanishes in each period: $z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0$ for $t > 1$ and $z_0(p_1) + y(p_1, p_2) = 0$ for $t = 1$, where z_0 is the excess demand of old consumers in period one. There are two types of equilibria in the economy. To distinguish between them we set $\mu = p_1'z_0(p_1)$. Iterated application of the equilibrium condition and Walras's law shows that $-p_t'y(p_t, p_{t+1}) = p_{t+1}'z(p_t, p_{t+1}) = \mu$ at all times. Thus μ is the fixed nominal savings of young people in each period or, equivalently, the fixed stock of fiat money (at least if $\mu > 0$). If $\mu = 0$, we call this a <u>real</u> path, if $\mu \neq 0$, we call it a <u>nominal</u> path.

A steady state of the economy is a relative price vector p and inflation factor γ such that $p_t = \gamma^t p$ is an equilibrium of the economy. The steady state rate of interest is $1/\gamma - 1$. In the remainder of this section we summarize some general results from Kehoe and Levine (1982a). If a steady state is nominal, then Walras's law and the equilibrium condition imply that

$\gamma = 1$. Conversely, if a steady state is real, then generically $\gamma \neq 1$. We focus on the behavior of paths near a steady state. In particular, we ask how many equilibrium paths converge to the steady state.

The stable manifold theorem from the theory of dynamical systems described in Irwin (1980) implies that generically this question can be answered by linearizing the equilibrium conditions. Making use of the fact that derivatives of excess demand are homogenous of degree minus one, we can write the linearized system near a steady state (p, γ) as

$$(2.1) \qquad D_2y\, p_{t+1} + (D_1y + \gamma D_2z)p_t + \gamma D_1z\, p_{t-1} = 0\ , \quad t > 1$$

$$(2.2) \qquad D_2y\, p_2 + (D_1y + Dz_0)p_1 = Dz_0\, p$$

where D_1y is, for example, the matrix of partial derivatives of y with respect to its first vector of arguments and where all derivatives are evaluated at $(p, \gamma p)$.

In the generic case D_2y is non-singular, and 2.1 can be solved to yield a second order difference equation. Define the matrix $R(\phi)$ by the rule

$$(2.3) \qquad R(\phi) = D_2y\, \phi^2 + (D_1y + \gamma D_2z)\phi + \gamma D_1z.$$

The characteristic values of the system are the roots of the equation $\det R(\phi_i) = 0$, and, if the vectors f_i satisfy $R(\phi_i)f_i = 0$ and $f_i \neq 0$, the characteristic vectors of the system are $(f_i, \phi_i f_i)$. Homogeneity implies that $\phi = \gamma$ is a root of R with $R(\gamma)p = 0$ where p is the steady state vector

of relative prices. Generically, this is the only root on the circle of radius γ in the complex plane.

We consider a nominal steady state with $\mu \neq 0$ and $\gamma = 1$ first. Assume $p_1' z_0(p_1) = \mu$ for all p_1, so that the initial nominal savings of old people are independent of prices. The initial prices p_1 and p_2 must satisfy $p_1' y(p_1, p_2) = -\mu$, which can be linearized as

$$(2.4) \qquad p' D_2 y \; p_2 + (y' + p' D_1 y) p_1 = -\mu .$$

This defines a $2n - 1$ dimensional subspace of the initial conditions (p_1, p_2), which is invariant since μ is constant on paths. It contains all $2n - 1$ characteristic vectors of the system except the vector (p, p) that corresponds to the root $\gamma = 1$. Let n^s be the number of roots inside the unit circle. The corresponding n^s dimensional subspace spanned by characteristic vectors is the space of initial conditions (p_1, p_2) that yield paths that converge to the steady state. Condition 2.2 defines an n dimensional subspace of the $2n - 1$ dimensional space of vectors that satisfy 2.4. The intersection of the two spaces generically has dimension $n + n^s - 2n - 1 = n^s - n + 1$. Thus, generically, there is an $n^s - n + 1$ dimensional set of equilibria. If $n^s < n - 1$, the set is empty; if $n^s = n - 1$, the equilibrium is unique.

In real steady states the price level is indeterminate, so we work with prices in a $2n - 1$ dimensional space of normalized prices, throwing out the characteristic value γ associated with characteristic vector $(p, \gamma p)$. It can be shown that the condition for stability in this lower dimensional system is that characteristic values be less than γ in modulus. Since we consider only initial conditions with $\mu = 0$, condition 2.4 further reduces the dimension of

the system to $2n - 2$. It can be shown using Walras's law that the eigenvalue thrown out in this reduction is equal to 1 and that this root governs the behavior of paths with nominal initial conditions near a real steady state: If $\gamma > 1$, then asymptotically money does not matter; if $\gamma < 1$, then initial conditions with valued money cannot yield equilibrium price paths that approach the steady state. Let the number of remaining eigenvalues that lie inside the circle of radius γ be $\bar{n}^s$. The initial condition 2.2 defines an $n - 1$ dimensional space and thus the dimension of equilibria that converge to the real steady state is $\bar{n}^s + (n - 1) - (2n - 2) = \bar{n}^s - n + 1$.

3. IMPLICATIONS OF A SINGLE COBB-DOUGLAS CONSUMER

Kehoe and Levine (1982c) prove that there are robust examples of economies with any value of $0 < n^s < 2n - 1$ and $0 < \bar{n}^s < 2n - 2$. A case of particular interest is when the characteristic values split, that is, $n - 1$ lie inside, and $n - 1$ outside, the circle of radius γ. In this case $\bar{n}^s = n - 1$, which implies that real paths are unique, and $n - 1 < n^s < n$, which implies that nominal paths are either unique or have a single dimension of indeterminacy. In the extreme case where $n - 1$ values lie outside the circle of radius γ and $n - 1$ values are exactly equal to zero, we say that the system splits exactly. In this case $n - 1$ prices are thrown out as unstable and $n - 1$ jump directly to their steady state value. The dynamical system really only involves two prices: one price for future consumption and one for current consumption. Thus, a system that splits exactly behaves like a model with one good in each period. In the real case we get to throw out one of these prices as corresponding to paths with valued fiat money, and we see that the system jumps directly to the steady state. In the nominal case there is one eigenvalue that is not determined. If it lies inside the unit

circle, there is one dimension of indeterminacy, but it is always possible to jump right to the steady state. If it lies outside the unit circle, there is a uniquely determined path going to, but not generally equal to, the steady state.

The Balasko-Shell result in our context states that if there is a single Cobb-Douglas consumer in each generation the system splits. Suppose the representative consumer has utility $u(y, z)$ for net trades. The consumer maximizes $u(y, z)$ subject to the budget constraint $p_t'y + p_{t+1}'z = 0$. Assuming that the utility function has all the necessary properties, we can characterize the solution to this problem by the usual first order conditions:

$$\text{(3.1)} \qquad D_1u - \lambda p_t' = 0$$

$$D_2u - \lambda p_{t+1}' = 0$$

$$p_t'y + p_{t+1}'z = 0$$

for some $\lambda > 0$. Using the implicit function theorem, we can compute the partial derivatives of y and z by differentiating 3.1. For example, after some tedious algebra we find that

$$\text{(3.2)} \qquad D_2y = -D_{11}^2u^{-1}(D_{12}^2u\ A(\lambda I - \tfrac{1}{b}cd') + \tfrac{1}{b}p_td')$$

where

$$(3.3) \qquad A = (D^2_{22}u - D^2_{21}u \; D^2_{11}u^{-1} \; D^2_{12}u)^{-1}$$

$$b = \begin{bmatrix} p'_t & p'_{t+1} \end{bmatrix} \begin{bmatrix} D^2_{11}u & D^2_{12}u \\ D^2_{21}u & D^2_{22}u \end{bmatrix}^{-1} \begin{bmatrix} p_t \\ p_{t+1} \end{bmatrix}$$

$$c = p_{t+1} - D^2_{21}u \; D^2_{11}u \; p_t$$

$$d = \lambda Ac + z(p_t, \; p_{t+1}).$$

A similar expression can be computed for D_1z. If, as is the case for the log linear form of Cobb-Douglas utility, utility is additively separable between y and z, then $D^2_{12}u = 0$. This implies that both D_2y and D_1z have at most rank one.

We now demonstrate that the condition rank D_2y = rank $D_1z = 1$, together with some generic assumptions, implies that the characteristic values split exactly. Furthermore, if D_2y and D_1z are approximately the same as rank one matrices, then the characteristic values split, but not exactly. This shows why, although in general n^s and $\overline{n}^s$ can be anything, separable preferences and a representative consumer impose strong restrictions: With several consumers with non-identical separable preferences D_2y and D_1z are sums of different rank one matrices and, consequently, cannot be expected to be rank one matrices even approximately.

To analyze the system when rank D_2y = rank $D_1z = 1$ we re-examine the characteristic polynomial det $R(\phi) = 0$. Although the singularity of D_2y prevents us from explicitly solving to find a second order difference

equation, we can still define the characteristic roots of det $R(\phi) = 0$ and characteristic vectors $(f_i, \phi_i f_i)$ where $R(\phi_i)f_i = 0$. We make the generic assumption that det $R(\phi)$ is not identically zero. This means that $R(\phi)$ is a regular pencil of matrices as described in Gantmacher (1959). Since $D_1 z$ is singular, zero is obviously a characteristic root, and, since $D_1 z$ has rank one, there are $n - 1$ corresponding characteristic vectors $(f_i, 0)$ where the f_i span the null space of $D_1 z$. Consequently, at least $n - 1$ roots lie inside the unit circle.

Running the system backwards in time, we are led to consider the backwards characteristic matrix $\beta^2 R(\beta^{-1})$. Since $D_2 y$ has rank 1, this has $n - 1$ zero roots, and, by factoring both the forwards and backwards polynomials, we see that the degree of the polynomial det $R(\phi)$ is equal to $2n - (n - 1) = n + 1$. Thus there are $n - 1$ zero roots, plus two other roots, one of which is γ and, at a real steady state, the other of which is 1. There are also $n - 1$ missing roots that correspond to zero roots of the backwards polynomial, which are, in effect, equal to infinity.

Let us make the generic assumption that there are $n + 1$ linearly independent characteristic vectors $(f_i, \phi_i f_i)$ with characteristic values ϕ_i. Let q_t be an $n + 1$ vector and consider the linear system

$$(3.4) \qquad q_{t+1} = \mathrm{diag}(\phi_i) q_t.$$

If we recover ordinary prices by the formula

$$(3.5) \qquad p_t = \Sigma_{i=1}^{n+1} q_t^i f_i$$

equation, we can still define the characteristic roots of det $R(\phi) = 0$ and characteristic vectors $(f_i, \phi_i f_i)$ where $R(\phi_i)f_i = 0$. We make the generic assumption that det $R(\phi)$ is not identically zero. This means that $R(\phi)$ is a regular pencil of matrices as described in Gantmacher (1959). Since $D_1 z$ is singular, zero is obviously a characteristic root, and, since $D_1 z$ has rank one, there are $n - 1$ corresponding characteristic vectors $(f_i, 0)$ where the f_i span the null space of $D_1 z$. Consequently, at least $n - 1$ roots lie inside the unit circle.

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If we recover ordinary prices by the formula

$$(3.5) \qquad p_t = \Sigma_{i=1}^{n+1} q_t^i f_i$$

it is easy to check that these prices satisfy the linearized system 2.1. If all solutions are of this form, then the system splits exactly since n - 1 roots are zero and n - 1 are "infinite" and do not permit solutions.

Let us also make the generic assumption that the n + 1 vectors $(f_i, \phi_i f_i)$ together with the n - 1 backwards characteristic vectors $(0, b_i)$, where the b_i span the null space of D_2y, span R^{2n} the space of initial conditions. What we must show is that the backwards characteristic vectors have zero weight in any solution; that is, there is no non-zero vector $x \in R^n$ such that $D_2y\, x = (D_1y + \lambda D_2z)b_i$. But generically the image of the n - 1 dimensional subspace spanned by the b_i under the linear operator $D_1y + \lambda D_2z$ does not intersect the one dimensional subspace spanned by D_2y except at the origin. We therefore have exact splitting.

Let D_2y and D_1z have rank one, and let $(D_1z^k, D_2z^k, D_1y^k, D_2y^k) \to (D_1z, D_2z, D_1y, D_2y)$. It follows from factoring the equation $\det R^k(\phi) = 0$ and $\det R(\phi) = 0$ that, as $k \to \infty$, the system splits, although not exactly.

4. INTERTEMPORAL SEPARABILITY AND RANK ONE MATRICES

We have seen that it is not Cobb-Douglas utility, nor even additive separability, that implies exact splitting, but rather that $\text{rank } D_1z = \text{rank } D_2y = 1$. We can expect this to hold in a neighborhood of a steady state only if it holds globally, and we have argued that there is never any reason to believe it holds with several consumers. With one consumer, however, if $u(y, z) = v(h(y), g(z))$, so utility depends only on an index of consumption each period, then

(4.1) $D_{12}^2 u = D_{12}^2 v \; Dh \; Dg'$

Furthermore, at the optimum $Dh = \lambda p_t$ and $Dg' = \lambda p'_{t+1}$. Therefore

(4.2) $D_{12}^2 u = \lambda^2 D_{12}^2 v \; p_t p'_{t+1}$.

Using 3.2, we see that

(4.3) $D_2 y = -D_{11} u^{-1} p_t (\lambda^2 D_{12}^2 v \; p'_{t+1} A(\lambda I - \frac{1}{b} cd') + \frac{1}{b} d')$

has rank one, and similarly for $D_1 z$. Consequently, intertemporally separable utility implies exact splitting.

5. GLOBAL ANALYSIS

Intertemporal separability implies that $D_2 y$ and $D_1 z$ have rank one for all values of (p_t, p_{t+1}). If we drop the stationarity assumption and analyze what happens locally near an arbitrary (not necessarily steady state) path, then the linearization 3.3 is still valid, although the non-zero eigenvalues ϕ_i^t now depend on t. This implies the linear system still depends on just two prices: the prices of current and of future consumption. Using the fact that this dependence on two prices is global, we can use the implicit function theorem to globally solve the non-linear system as $q_{t+1} = f_t(q_t)$ where $q_t \in R^2$. Original prices are recovered from a non-linear equation $p_t = g_t(q_t)$. Therefore, the non-stationary system acts globally like a two good economy, and this implies that, as Balasko and Shell originally proved,

exact splitting does not depend on a local analysis near the steady state of a stationary model. Notice, however, that the notion of non-exact splitting does not generalize to the non-stationary case. In particular, when eigenvalues are large but not infinite, there may correspond explosive paths for relative prices that nonetheless are ultimately bounded and are thus legitimate equilibria.

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